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A Mixed Phase Transition for a Hierarchical Spin Glass

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A proof of the existence of a mixed ferromagnetic (or antiferromagnetic)-spin-glass fixed point for an Ising spin-glass model on the diamond hierarchical lattice is given.

KEY WORDS: Phase transitions; spin glasses; renormalization group; hierarchical models.

1. INTRODUCTION

In this paper we study a class of spin-glass models consisting of Ising variables on a hierarchical diamond lattice with random next-neighbor interactions. By construction, the Migdal–Kadanoff renormalization transformation is exact.

Following Collet and Eckmann,[1] we view this problem as a problem of random variables and we want to study the evolution of the probability distribution of the couplings under the action of the renormalization group. We are interested in limit distributions that remain unchanged under a change of lattice scale. Collet and Eckmann have demonstrated the existence of such a limit in a space of bounded symmetric functions. Besides low- and high-temperature limit distributions, they found a nontrivial fixed point corresponding to a spin-glass transition. It was in fact shown that this model has zero magnetization and nonzero Edwards–Anderson order parameter at low temperature. However, its Parisi[2] overlap function has been shown to be trivial.[3]

The purpose of this paper is to extend the analysis of Ref. 1 to a larger class of models and to include nonsymmetric probability distributions of

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coupling constants. We prove the existence of another fixed point corresponding to a transition to a mixed ferromagnetic (antiferromagnetic)–spin-glass phase.

In Section 2 we describe the model and give a heuristic derivation of the phase diagram. The diagram was obtained in Ref. 3, but our results on the flow in the low-temperature phase are slightly different. In Section 3, a proof is given for the existence of the mixed fixed point. Some details of the proofs are left to the Appendix.

2. THE PHASE DIAGRAM

The model we want to study consists of Isinglike variables s_i localized on the sites i of a recursive lattice and interacting with nearest neighbors with a coupling constant ε_{ij}, which is a random variable. The lattice is constructed as follows (see Fig. 1). First we choose an integer $n \geqslant 1$. We start with two sites and a link connecting them. Call this graph L_0. Then we substitute the link by n new sites and connect each one of the new sites with the extremes of the old link. We obtain L_1. It has $2+n$ sites and $2n$ links. We iterate, i.e., given L_N, we obtain L_{N+1} by substituting for each link n new sites and $2n$ new links.

Now consider the system at a given level N. Associated with it is the energy of a configuration of spins $\{s\}$ and links $\{\varepsilon\}$,

$$H_N(\{s\}, \{\varepsilon\}) = \sum_{(i,j)} \varepsilon_{ij} s_i s_j$$

where the sum is over nearest neighbors.

We shall consider the ε_{ij} as independent, identically distributed random variables, with probability distribution $f(\varepsilon)$.

If we apply a renormalization group transformation by summing up

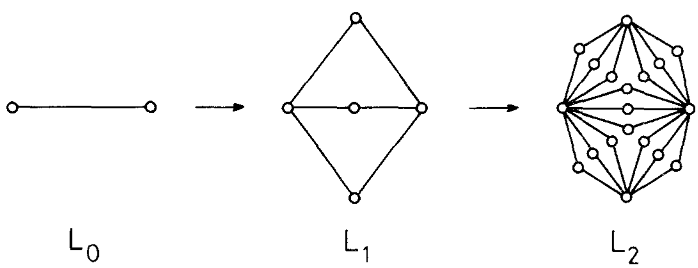

Fig. 1. The construction of the lattice for $n = 3$.

the spins created at level N, we end up with the same system but at the level $N-1$ and with new effective coupling constant $\hat{\varepsilon}$ given by

$$\hat{\varepsilon} = \sum_{i=1}^{n} \tilde{\varepsilon}_i \tag{1a}$$

$$\tilde{\varepsilon} = \operatorname{tgh}^{-1}(\operatorname{tgh} \varepsilon \operatorname{tgh} \varepsilon') \tag{1b}$$

The variables $\hat{\varepsilon}$ are again independent and identically distributed with probability distribution $\hat{f}$ (see Fig. 2). We are looking for distributions that are invariant under the transformation induced by (1).

Observe that the pure system is obtained by setting $f(\varepsilon) = \delta(\varepsilon - J)$, so in this case the fixed point distribution is $\delta(\varepsilon - J^*)$, with J^* given by

$$J^* = n \operatorname{tgh}^{-1}(\operatorname{tgh}^2 J^*)$$

which has a nontrivial solution for $n > 1$.

We next describe heuristically the behavior of the random variables under renormalization.

First observe that if n is very large, Eq. (1a) suggests that, by the central limit theorem, every bounded distribution is mapped on a Gaussian. So we will look for a Gaussian fixed distribution and we will parametrize our functions by a mean J and a variance σ. We will also consider the family of pure systems as the subspace $\sigma = 0$. With this in mind let us now study the evolution of the parameters J and σ under renormalization. We have

$$\hat{J} = n\tilde{J} = n \int T(x, y) f(x) f(y)\, dx\, dy = \hat{J}(J, \sigma^2) \tag{2a}$$

$$\hat{\sigma}^2 = n\tilde{\sigma}^2 = n \int [T(x, y) - \tilde{J}]^2 f(x) f(y)\, dx\, dy = \hat{\sigma}^2(J, \sigma^2) \tag{2b}$$

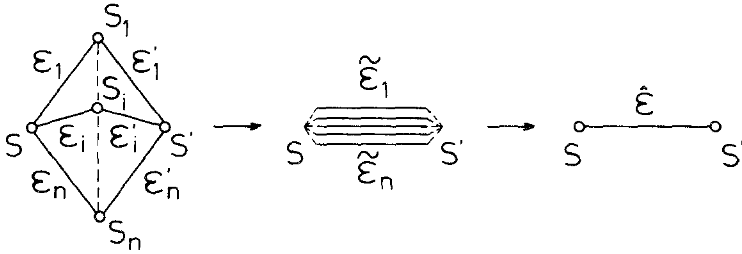

Fig. 2. The decimation of the spins s_i between s and s'.

with

$$T(x, y) = \mathrm{tgh}^{-1}(\mathrm{tgh}\, x\, \mathrm{tgh}\, y), \qquad f(x) \sim \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \frac{-(x-J)^2}{2\sigma^2}$$

If σ^2 and J are small, we obtain

$$\hat{J} \sim n(J^2 - 2J^2\sigma^2 + \cdots), \qquad \hat{\sigma}^2 \sim n(\sigma^4 + 2J^2\sigma^2 + \cdots)$$

This gives us four fixed points: (0,0), the high-temperature one; $(1/n, 0)$, the pure ferromagnetic one; $(0, 1/n)$, the pure spin-glass one; and $(1/n, 1/n)$, a mixed fixed point. We use the word mixed here to indicate the disordered *and* ferromagnetic character of the interactions. A demonstration of the existence for the pure spin-glass fixed point as well as the high- $(\sigma \sim 0)$ and low-temperature $(\sigma \sim \infty)$ behavior for $J = 0$ is given in Ref. 1. We give in the next section a proof of the existence of the mixed fixed point.

In the remainder of this section we investigate heuristically the behavior of the system in the low-temperature regions, that is, the flow of the points (J, σ^2) under Eqs. (2) if one or both of the components is big.

Consider first the case $J \gg 1$, $\sigma^2 \ll 1$. Then we obtain, by expanding $T(x, y)$ around $\tilde{J}_0 = \mathrm{tgh}^{-1}(\mathrm{tgh}^2 J)$,

$$\hat{J} \sim nJ, \qquad \hat{\sigma}^2 \sim n\sigma^2$$

This shows that we have a zero-temperature pure ferromagnetic fixed point $(\infty, 0)$, but it is unstable with respect to the disorder σ^2.

To analyze the other cases $(\sigma^2 \gg 1)$, let us rewite Eq. (2a) as

$$\hat{J} = n\left[\tilde{J}_0 - \frac{1}{2}\int_0^\infty |\ln(1 - t^2\,\mathrm{tgh}^2 x)| \frac{1}{(4\pi\sigma^2)^{1/2}} \exp -\frac{x^2}{4\sigma^2}\, dx\right]$$

where $t = \mathrm{tgh}\, 2J$. Then we have

$$\hat{J} \sim \begin{cases} nJ[1 - O(\sigma/J)] & \text{if} \quad \sigma/J \ll 1 \\ nJO(J/\sigma) & \text{if} \quad J/\sigma \ll 1 \end{cases}$$

For the variance we always have for low temperature

$$\hat{\sigma}^2 \sim n\sigma^2$$

This means that if we start with a low-temperature system with almost all bounds ferromagnetic, but with a small broadening around J $(J \gg 1 \gg \sigma)$, this system becomes more and more disordered (σ increases) under the

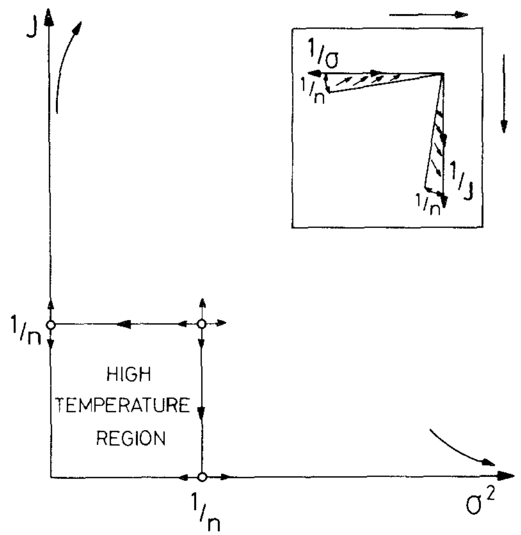

Fig. 3. The phase diagram. Phase boundaries are indicated qualitatively.

action of the renormalization group, ending in a fixed point (∞, ∞), but remains essentially ferromagnetic.

On the other hand, if we begin with a system that is more disordered than ferromagnetic $(J \ll \sigma)$, then we will end up with a pure spin-glass system in a fixed point $(0, \infty)$ (Fig. 3). The case $J = 0$, $\sigma \gg 1$ was analyzed in Ref. 1.

3. THE MIXED FIXED POINT

The idea is to show that the operator described above, which corresponds to the action of the renormalization group on the space of the probability distributions for the couplings, has a fixed point in a suitably chosen space.

Since we expect the fixed distribution to be almost a Gaussian, it is reasonable to parametrize our space by a mean J, a squared variance σ^2, and a function ϕ. This function represents the non-Gaussian part of the probability distribution. Moreover, we expct the nontrivial fixed point to have a mean J and a squared variance σ^2 of order $1/n$, so we naturally define dimensionless parameters p and s, which corresponds to nJ and $n\sigma^2$, respectively.

We start with a triplet (p, s, ϕ), which is, in a sense to be made precise, close to $(1, 1, 0)$. To this triplet is associated a function f, which corresponds to the probability distribution of the interaction ε at a given level of the hierarchy (the ε_i and the ε_i' of Fig. 1). The function associated to $(1, 1, 0)$ is a Gaussian with mean $1/n$ and squared variance $1/n$. Then we sum over the s_i spins. This corresponds to associating to the function f a new function $\tilde{f}$, which stands for the probability distribution of the $\tilde{\varepsilon}_i$. This step introduces a highly nonlinear and non-Gaussian effect into the distribution. We call $\tilde{J}$ and $\tilde{\sigma}$ the mean and the variance of $\tilde{f}$. The next step is to sum over the n independent, identically distributed variables $\tilde{\varepsilon}_i$ to obtain the new effective coupling $\hat{\varepsilon}$; this is achieved by taking the convolution product of $\tilde{f}$ a total of n times to obtain $\hat{f}$, the distribution of the $\hat{\varepsilon}$. This clearly has mean $\hat{J} = n\tilde{J}$ and variance $\hat{\sigma}^2 = n\tilde{\sigma}^2$. If n is sufficiently large, $\hat{f}$ is nearly Gaussian and is reparametrized as $(\hat{p}, \hat{s}, \hat{\phi})$, which should also be close to $(1, 1, 0)$.

3.1. Definitions and Notations

Let us now introduce the main definitions

$$H = \left\{ (p, s, \phi) \mid p, s \in \mathbf{R}; \phi \in \mathbf{L}_\infty(\mathbf{R}); |(p, s, \phi)| < \infty; \right.$$

$$\left. \int \phi(x)\, dx = 0, \int x\phi(x)\, dx = 0, \int x^2 \phi(x)\, dx = 0 \right\}$$

where

$$|(p, s, \phi)| \equiv |p| + |s| + |\phi|$$

$$|\phi| \equiv \|\phi\|_\infty + \|x^3\phi\|_2 + \|x^4\phi\|_2$$

with

$$\|x^k\phi\|_q = \left(\int |x^k\phi(x)|^q \right)^{1/q}$$

To each $(p, s, \phi) \in \mathbf{H}$ with $s > 0$ we associate a function f by

$$f(x) = (n/s)^{1/2}\, \phi(n^{1/2}x) + h_{p/n, s/n}(x)$$

where $h_{J,\sigma^2}(x)$ denotes the Gaussian

$$(2\pi\sigma^2)^{-1/2} \exp - [(x - J)^2/2\sigma^2]$$

(Although it is not necessary, we will also consider $p \geqslant 0$.)

We define

$$\mathbf{H}_1 \equiv \{ f \mid |f|_{H_1} \equiv n^{-1/2} \|f\|_\infty + n^{5/4} \|x^3 f\|_2 + n^{7/4} \|x^4 f\|_2 < \infty \}$$

The relation $S_1(p, s, \phi) = f$ defines the operator S_1: $\mathbf{H} \to \mathbf{H}_1$. Denote the action of the tangent map by

$$DS_{1_{(p,s,\phi)}}(r, t, \psi) = g$$

with components

$$DS_{1_{(p,s,\phi)}}(r, 0, 0) = g_p$$
$$DS_{1_{(p,s,\phi)}}(0, t, 0) = g_s$$
$$DS_{1_{(p,s,\phi)}}(0, 0, \psi) = g_\phi$$

In general we shall write g_i, where i stands for p, s, or ϕ.

Consider next for f, $g \in \mathbf{H}_1$

$$A(f, g)(x) = (1 - \mathrm{tgh}^2 x) \int_\Omega \frac{dy}{|y| (1 - y^2)[1 - (\mathrm{tgh}^2 x)/y^2]} \times f(\mathrm{tgh}^{-1} y) f\left(\mathrm{tgh}^{-1} \frac{\mathrm{tgh}\, x}{y}\right)$$

with $\Omega = \{ y, |\mathrm{tgh}\, x| < |y| < 1 \}$. Note that $A(f, g) = A(g, f)$.

As we shall see, A maps$(\mathbf{H}_1 \times \mathbf{H}_1)$ to $\mathbf{K}$ defined as

$$\mathbf{K} \equiv \{\tilde{f} |\ |\tilde{f}|_{\mathbf{K}} < \infty \}$$

with

$$|\tilde{f}|_{\mathbf{K}} = \|\tilde{f}\|_1 + n^{-1/2}(\ln n)^{-1} \|\tilde{f}\|_2 + n^{5/2} \|x^3\tilde{f}\|_1 + n^{7/2} \|x^4\tilde{f}\|_1$$

So we define the map S: $\mathbf{H}_1 \to \mathbf{K}$ by

$$(Sf)(x) = \tilde{f}(x) = A(f, f)$$

The tangent map is given by

$$DS_f g = \tilde{g} = 2A(f, g)$$

Next define the operation T: $\mathbf{K} \to \mathbf{H}_1$ by

$$(T\tilde{f})(x) = \hat{f}(x) \equiv \tilde{f}^{*n}(x)$$

which is the n-fold convolution product of $\tilde{f}$. We denote $\hat{g}(x) = DT_{\tilde{f}}\tilde{g}$.

The last operation T_1: $\mathbf{H}_1 \to \mathbf{H}$ is defined by

$$T_1 \hat{f} = (\hat{p}, \hat{s}, \hat{\phi})$$

with

$$\hat{p} = n \int x\hat{f}(x)\,dx$$

$$\hat{s} = n\left[\int x^2\hat{f}(x)\,dx - (\hat{p}/n)^2\right]$$

$$\hat{\phi}(n^{1/2}x) = (\hat{s}/n)^{1/2}[\hat{f}(x) - h_{\hat{p}/n,\hat{s}/n}(x)]$$

Observe that $T_1 = S_1^{-1}$.

Let $M = T_1 TSS_1$. What we want to show is that M has a fixed point near $(1, 1, 0)$. This means that the "renormalization group operator" has a fixed distribution that is almost a Gaussian with mean $1/n$ and variance $1/n^{1/2}$. We also want to show that these are the only relevant variables. This should imply that the tangent map DM is almost diagonal with respect to these variables. It turns out that two off-diagonal elements are big with the norm on **H** defined above. However, since the determinant of DM is almost equal to the product of the diagonal elements, we can make a scale change, which means we can consider an equivalent norm on **H**,

$$|(p, s, \phi)|_{\mathbf{H}} = \frac{1}{n^{1/4}\ln n}|p| + \frac{1}{\ln n}|s| + |\phi|$$

We state our main results with this equivalent norm, although in all the intermediate steps only the norm introduced before is needed. In fact we can consider the action of DM on each of the subspaces of the tangent space separately.

Theorem 1. Let

$$\mathbf{B} = \left\{(p, s, \phi) \,|\, |(p-1, s-1, \phi)|_{\mathbf{H}} \leqslant \frac{(\ln n)^3}{n}\right\}$$

The operator M maps **B** to **H** and has a unique fixed point in **B**. The map DM has two simple eigenvalues at $2 + O(n^{-1/2})$ and the remainder of the spectrum strictly inside the unit disk.

To demonstrate Theorem 1 by the modified Newton's method we need the following.

Theorem 2:

1. $|M(1, 0, 0) - (1, 0, 0)|_{\mathbf{H}} \leqslant (\ln n)/n$.

2. M is defined as a map from $\mathbf{B}$ to $\mathbf{H}$ and $DM_{(p,s,\phi)}$ is a 3×3 "matrix"

$$DM = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

whose elements satisfy

$$A = 2 + O(n^{-1/2}), \qquad E = 2 + O(n^{-3/4})$$

$$|B(0, t, 0)|_{\mathbf{H}} \leqslant (1/n)\,|(0, t, 0)|_{\mathbf{H}}$$

$$|C(0, 0, \psi)|_{\mathbf{H}} \leqslant (1/\ln n)\;|(0, 0, \psi)|_{\mathbf{H}}$$

$$|D(r, 0, 0)|_{\mathbf{H}} \leqslant (1/n^{1/4})\;|(r, 0, 0)|_{\mathbf{H}}$$

$$|F(0, 0, \psi)|_{\mathbf{H}} \leqslant (1/\ln n)\;|(0, 0, \psi)|_{\mathbf{H}}$$

$$|G(r, 0, 0)|_{\mathbf{H}} \leqslant [(\ln^2 n)/n^{1/4}]\;|(r, 0, 0)|_{\mathbf{H}}$$

$$|H(0, t, 0)|_{\mathbf{H}} \leqslant [(\ln^2 n)/n^{1/2}]\;|(0, t, 0)|_{\mathbf{H}}$$

$$|I(0, 0, \psi)|_{\mathbf{H}} \leqslant [(\ln n)/n^{1/4}]\;|(0, 0, \psi)|_{\mathbf{H}}$$

We will now describe all the steps, propositions, and lemmas that lead to Theorem 2. Part 1 will follow after little effort from Lemma 4. The elements A and E of the matrix DM of part 2 are bounded in Corollary 7. Bounds on B, C, D, and F follow from Lemma 5 and those on G, H, and I immediately from Proposition 12.2. Most of the proofs are essentially the same as in the symmetric case, so we refer to the original paper.[1]

3.2. The Operator S_1

We have $S_1(p, s, \phi) = f$ and

$$f(x) = h_{p/n, s/n}(x) + (n/s)^{1/2}\,\phi(n^{1/2}x)$$

From the definitions it is obvious that $|S_1(p, s, \phi)|_{\mathbf{H}_1} \leqslant O(1)$ for $(p, s, \phi) \in \mathbf{B}$, so

$$\|S_1\|_{\mathbf{B} \to \mathbf{H}_1} \leqslant O(1)$$

Observe that for $(p, s, \phi) \in \mathbf{B}$

$$|(p, s, \phi)| = 1 + O(n^{-1/2})$$

We also have

$$DS_{1_{(p,s,\phi)}}(r, t, \psi)(x) \equiv g(x) \equiv g_p(x) + g_s(x) + g_\phi(x)$$

with

$$g_p(x) = r\,\frac{1}{s}\left[\left(x - \frac{p}{n}\right) h_{p/n,s/n}(x)\right]$$

$$g_s(x) = t\,\frac{1}{2s}\left[\frac{n}{s}\left(x - \frac{p}{n}\right)^2 h_{p/n,s/n}(x) - f(x)\right]$$

$$g_\phi(x) = \left(\frac{n}{s}\right)^{1/2} \psi(n^{1/2}x)$$

It is easy to show that for $(p, s, \phi) \in \mathbf{B}$

$$|DS_{1_{(p,s,\phi)}}(r, t, \psi)|_{\mathbf{H}_1} \leqslant O(1)\,|(r, t, \psi)|_{\mathbf{H}}$$

so

$$\|DS_{1_{(p,s,\phi)}}\|_{\mathbf{H} \to \mathbf{H}_1} \leqslant O(1)$$

Further, notice that

$$\int g_p(x)\,dx = 0, \qquad \int \left(x - \frac{p}{n}\right) g_p(x)\,dx = \frac{r}{n}$$

$$\int \left(x - \frac{p}{n}\right)^2 g_p(x)\,dx = 0$$

$$\int g_s(x)\,dx = 0, \qquad \int \left(x - \frac{p}{n}\right) g_s(x)\,dx = 0$$

$$\int \left(x - \frac{p}{n}\right)^2 g_s(x)\,dx = \frac{t}{n}$$

$$\int g_\phi(x)\,dx = \int x g_\phi(x)\,dx = \int x^2 g_\phi(x)\,dx = 0$$

3.3. The Operator S

We have

$$Sf = \tilde{f} = A(f, f)$$

$$DS_f g = \tilde{g} = 2A(f, g)$$

If $f \in S_1 \mathbf{B} \subset \mathbf{H}_1$, $g \in \mathbf{H}_1$, we have from Lemmas A2 and A4 of the Appendix

$$\|A(f, g)\|_1 \leqslant \|g\|_1$$
$$\|A(f, g)\|_2 \leqslant O(1)(\|g\|_1 \|f\|_\infty + \|g\|_\infty \ln \|f\|_\infty)$$
$$\|x^3 A(f, g)\|_1 \leqslant O(1) \|xf^3\|_2 \|x^3 g\|_2$$
$$\|x^4 A(f, g)\|_1 \leqslant O(1) \|xf^4\|_2 \|xg^4\|_2$$

Since we know that $|f|_{\mathbf{H}_1} \leqslant O(1)$, we have from the definitions of $|\cdot|_{\mathbf{H}_1}$ and $|\cdot|_{\mathbf{K}}$ and the estimates above

$$|SS_1(p, s, \phi)|_{\mathbf{K}} \leqslant O(1)$$
$$|DSS_{1_{(p,s,\phi)}}(r, t, \psi)|_{\mathbf{K}} \leqslant O(1) \ |(r, t, \psi)|$$

So

$$\|SS_1\|_{\mathbf{B} \to \mathbf{K}} \leqslant O(1)$$
$$\|DSS_{1_{(p,s,\phi)}}\|_{\mathbf{H} \to \mathbf{K}} \leqslant O(1)$$

Moreover, a good knowledge of the moments of $\tilde{f}$ and $\tilde{g}$ is also required. If $(p, s, \phi) \in \mathbf{B}$ and $(r, t, \psi) \in \mathbf{H}$, then for $\tilde{f} = SS_1(p, s, \phi)$ and $\tilde{g} = DSS_{1_{(p,s,\phi)}}(r, t, \psi)$ we have the following lemmas.

Lemma 4. The mean and the variance of $\tilde{f}$ are given by

$$\tilde{J} = \int_{-\infty}^{+\infty} x\tilde{f}(x)\, dx = p^2/n^2 + O(n^{1/2}/n^3)$$

$$\tilde{\sigma}^2 = \int_{-\infty}^{+\infty} (x - \tilde{J})^2\, \tilde{f}(x)\, dx = s^2/n^2 + O(n^{1/4}/n^3)$$

Moreover, the third moment of $\tilde{f}$ is bounded as

$$\tilde{\mu}_3 = \left| \int_{-\infty}^{+\infty} (x - \tilde{J})^3\, \tilde{f}(x)\, dx \right| \leqslant O(1/n^4)$$

Lemma 5. The moments of $\tilde{g}_i$ $(i = p, s, \phi)$ satisfy

$$\int_{-\infty}^{+\infty} x\tilde{g}_p(x)\, dx = rp/n^2 + |r| \ O(1/n^{5/2})$$

$$\left| \int_{-\infty}^{+\infty} x\tilde{g}_s(x)\, dx \right| \leqslant |t| \ O(n^{1/4}/n^3)$$

$$\left|\int_{-\infty}^{+\infty} \tilde{g}_\phi(x)\,dx\right| \leqslant |\psi|\; O(n^{1/4}/n^3)$$

$$\left|\int_{-\infty}^{+\infty} x^2\tilde{g}_p(x)\,dx\right| \leqslant |r|\; O(1/n^{5/2})$$

$$\int_{-\infty}^{+\infty} x^2\tilde{g}_2(x)\,dx \leqslant t/n^2 + |t|\; O(n^{1/4}/n^3)$$

$$\left|\int_{-\infty}^{+\infty} x^2\tilde{g}_\phi(x)\,dx\right| \leqslant |\psi|\; O(1/n^2)$$

$$\left|\int_{-\infty}^{+\infty} x^3\tilde{g}_p(x)\,dx\right| \leqslant |r|\; O(1/n^4)$$

$$\left|\int_{-\infty}^{+\infty} x^3\tilde{g}_s(x)\,dx\right| \leqslant |t|\; O(1/n^4)$$

$$\left|\int_{-\infty}^{+\infty} x^3\tilde{g}_\phi(x)\,dx\right| \leqslant |\psi|\; O(1/n^3 n^{1/2})$$

The proof of these lemmas is relegated to Appendix A2.

On the other hand, by making estimates similar to Lemma A4, it is easy to obtain better bounds for the fourth moment as well.

Lemma 6

$$\int_{-\infty}^{+\infty} x^4\,|\tilde{f}(x)|\,dx \leqslant O(1/n^4)$$

$$\int_{-\infty}^{+\infty} x^4\,|\tilde{g}_p(x)|\,dx \leqslant |r|\; O(1/n^4 n^{1/2})$$

$$\int_{-\infty}^{+\infty} x^4\,|\tilde{g}_s(x)|\,dx \leqslant |t|\; O(1/n^4)$$

$$\int_{-\infty}^{+\infty} x^4\,|\tilde{g}_\phi(x)|\,dx \leqslant |\psi|\; O(n^{1/4}/n^4)$$

At this point we have all that we need to estimate the elements of the first two lines of the "matrix" $DM_{(p,s,\phi)}$ of Theorem 2, that is, the variation of $\hat{p}$ and $\hat{s}$ as functions of p, s, and ϕ. These estimates follow as corollaries of Lemma 5.

Corollary 7. The two first diagonal elements of $DM_{(p,s,\phi)}$, A and E, are given by

$$A = 2 + O(1/n^{1/2}); \qquad E = 2 + O(1/n^{3/4})$$

Proof. Since

$$A = 2n^2 \int_{-\infty}^{+\infty} x[DSS_{1_{(p,s,\phi)}}(1, 0, 0)](x)\, dx = n^2 \int_{-\infty}^{+\infty} x\tilde{g}_p(x)\, dx$$

$$\begin{aligned} E &= 2n^2 \int_{-\infty}^{+\infty} x^2[DSS_{1_{(p,s,\phi)}}(0, 1, 0)](x)\, dx \\ &\quad - 2n^2 \int_{-\infty}^{+\infty} x[S_{1_{(p,s,\phi)}}(1, 0, 0)(x)]\, dx \\ &\quad \times \int_{-\infty}^{+\infty} x[DSS_{1_{(p,s,\phi)}}(1, 0, 0)(x)]\, dx \\ &= 2n^2 \int_{-\infty}^{+\infty} x^2 \tilde{g}_s(x)\, dx - 2n\tilde{J}A \end{aligned}$$

The assertion is obvious from Lemma 5.

The bounds on B, C, D, and F follow exactly in the same way from Lemma 5.

We also need the following properties of $\tilde{f}$:

Lemma 8. Let

$$\tilde{f} \in SS_1 \mathbf{B}, \qquad f_h = S_1(p, s, 0) = h_{p/n, s/n}$$
$$f_\phi(x) = (n/s)^{1/2}\, \phi(n^{1/2} x)$$

We write $\tilde{f}$ as $\tilde{f} = \tilde{f}_h + \tilde{w}$, where

$$\tilde{f}_h = A(f_h, f_n) = SS_1(p, s, 0)$$
$$\tilde{w} = 2A(f_h, f_\phi) + A(f_\phi, f_\phi)$$

Then we have

(a) $\tilde{f}_h > 0$

(b) $\int_{-\infty}^{+\infty} \tilde{f}_h(x)\, dx = \|\tilde{f}_h\|_1 = 1, \qquad \|\tilde{f}_h\|_2 \leqslant O(n^{1/2} \ln n)$

(c) $\|\tilde{w}\|_1 \leqslant O(n^{-3/4})$

This is a trivial variation of Remark 6.8.[1]

3.4. The Operator $T_1 T$

This operator corresponds to taking the sum of the n intermediate spins, that is, we want to estimate the sum of the n independent, identically distributed variables $\tilde{\varepsilon}$ each one with $\tilde{f} \in SS_1\mathbf{B} \subset \mathbf{K}$ as probability distribution. The operator $T_1 T\tilde{f}$ has three components:

$$\hat{p} = n^2 \int x\tilde{f}(x)\,dx = 1 + O(n^{1/4}/n)$$

$$\hat{s} = n^2 \int x^2\tilde{f}(x)\,dx - \hat{p}^2/n = 1 + O(n^{1/4}/n)$$

$$\hat{\phi}(x) = (\hat{s}/n)^{1/2}[\tilde{f}^{*n}(n^{-1/2}x) - h_{\hat{p}/n,\hat{s}/n}(x)]$$

We want to estimate $|\hat{\phi}|$. For this, we are going to make use of central limit theorems. These estimates are on the Fourier transforms. We shall also deal with centered functions.

Let us denote

$$\hat{f}^0(x) = \tilde{f}(x + \tilde{J}) = \tilde{f}\left(x + \frac{\hat{p}}{n^2}\right)$$

$v = F\hat{f}^0$ its Fourier transform, and

$$\hat{v}_n(t) = [v(n^{1/2}t)]^n$$

With this notation the object to study is

$$\hat{v}_n(t) - \exp(-\hat{s}t^2/2) = F[(1/\hat{s})^{1/2}\,\hat{\phi}(x + \hat{p}/n^{1/2})]$$

In order to estimate the elements of the third row of the matrix $DM_{(p,s,\phi)}$, that is, the variations $\partial_i\hat{\phi}$ of the rest function ϕ, let us write

$$\tilde{g}_i^0(x) = \tilde{g}_i(x + \hat{p}/n^{1/2}), \qquad \omega_i = F\tilde{g}_i^0$$
$$\hat{\omega}_n^i(t) = [v(n^{1/2}t)]^{n-1}\,\omega_i(n^{1/2}t)$$

We shall bound the $\partial_i\hat{\phi}$ by observing that for $i = p, s, \phi$

$$\partial_i\hat{f}(x) = \partial_i[h_{\hat{p}/n,\hat{s}/n}(x)] + \partial_i\left[\left(\frac{n}{\hat{s}}\right)^{1/2}\hat{\phi}(n^{1/2}x)\right]$$

and

$$\left(\frac{n}{\hat{s}}\right)^{1/2}\partial_i\phi(n^{1/2}x)$$
$$= \hat{g}_i(x)\left[\partial_{\hat{s}}h_{\hat{p}/n,\hat{s}/n}(x) - \frac{1}{2\hat{s}}\left(\frac{n}{\hat{s}}\right)^{1/2}\hat{\phi}(n^{1/2}x)\right]\partial_i\hat{s}$$
$$-[\partial_{\hat{p}}h_{\hat{p}/n,\hat{s}/n}(x)]\,\partial_i\hat{p}$$

But

$$\partial_i \hat{p} = n^2 \int x\tilde{g}_i^0(x)\,dx, \qquad \partial_i \hat{s} = n^2 \int x^2 \tilde{g}_i^0(x)\,dx$$

and

$$\partial_{\hat{p}} h_{\hat{p}/n,\hat{s}/n}(x) = \frac{-1}{s}\left(x - \frac{\hat{p}}{n}\right) h_{\hat{p}/n,\hat{s}/n}(x)$$

$$\partial_{\hat{s}} h_{\hat{p}/n,\hat{s}/n}(x) = \left[\frac{1}{2\hat{s}}\frac{n}{\hat{s}}\left(x - \frac{\hat{p}}{n}\right)^2 - 1\right] h_{\hat{p}/n,\hat{s}/n}(x)$$

Then we can write

$$\begin{aligned}
&\left(\frac{n}{\hat{s}}\right)^{1/2} \partial_i \hat{\phi}\left(n^{1/2}x + \frac{\hat{p}}{n^{1/2}}\right) \\
&\quad = \hat{g}_i^0(x) + \frac{n^2}{\hat{s}} x h_{0,s}(x)\left[\int x\tilde{g}_i^0(x)\,dx\right] \\
&\qquad - \frac{n^2}{2\hat{s}}\left(\frac{n}{\hat{s}}x^2 - 1\right) h_{0,\hat{s}/n}(x)\left[\int x^2\tilde{g}_i^0(x)\,dx\right] \\
&\qquad + \frac{1}{2\hat{s}}\left(\frac{n}{\hat{s}}\right)^{1/2} \hat{\phi}\left(n^{1/2}x + \frac{\hat{p}}{n^{1/2}}\right)
\end{aligned}$$

So

$$\begin{aligned}
&\left(\frac{n}{\hat{s}}\right)^{1/2} \partial_i \hat{\phi}\left(n^{1/2}x + \frac{\hat{p}}{n^{1/2}}\right) \\
&\quad = \hat{g}_i^0(x) + \frac{n^2}{\hat{s}} x h_{0,\hat{s}/n}(x)\left[\int x\tilde{g}_i^0(x)\right] \\
&\qquad - \frac{n^2}{2\hat{s}}\left(\frac{n}{\hat{s}}x^2 - 1\right) h_{0,\hat{s}/n}(x)\left[\int x^2\tilde{g}_i^0(x)\right] \\
&\qquad + \frac{1}{2\hat{s}}\left(\frac{n}{\hat{s}}\right)^{1/2} \hat{\phi}\left(n^{1/2}x + \frac{\hat{p}}{n^{1/2}}\right)
\end{aligned}$$

with

$$\hat{g}_i^0(x) = n(\tilde{f}^0)^{*n-1} * \tilde{g}_i^0(n^{-1/2}x)$$

If we define

$$\mathring{g}_i(x) = n^{-3/2}\hat{g}_i^0(n^{-1/2}x)$$

we have

$$\frac{1}{n}\frac{1}{\hat{s}^{1/2}}\partial_i\hat{\phi}\left(x+\frac{\hat{p}}{n^{1/2}}\right)-\frac{1}{n}\frac{1}{2\hat{s}^{3/2}}\hat{\phi}\left(x+\frac{\hat{p}}{n^{1/2}}\right)$$
$$=\mathring{g}(x)-n^{1/2}\frac{n}{\hat{s}}h_{0,\hat{s}}(x)-\frac{n}{2}\left(\frac{x^2}{\hat{s}^2}-\frac{1}{\hat{s}}\right)h_{0,\hat{s}}(x)$$

We want to estimate the Fourier transform of the right-hand side of the above equation,

$$\hat{\omega}_n^i(t)-[\exp(-\hat{s}t^2/2)][\omega_i'(0)+\omega_i''(0)\,nt^2/2]$$

Define

$$\rho_3=\frac{1}{\tilde{\sigma}^3}\int x^3\tilde{f}^0(x)\,dx, \qquad \rho_4=\frac{1}{\tilde{\sigma}^4}\int x^4\,|\tilde{f}^0(x)|\,dx$$

where

$$\tilde{\sigma}^2=\int x^2\tilde{f}^0(x)\,dx$$

Note that $|\rho_3|\leqslant O)1/n)$ and $1\leqslant\rho_4\leqslant O(1)$ (Lemmas 5 and 6). The fact that $|\rho_3|$ is only $O(1/n)$ and not $O(1)$ is very important and makes it possible to obtain the same bounds as in the case of symmetric distributions. It is in fact probably true that the moments μ_p of fixed distributions for diamond lattices should satisfy the property that μ_{2p-1} is of the same order as μ_{2p}.[4]

At this point it is convenient to introduce some more notations in order to state the results in a more compact form. We call $|e_p|=|r|$, $|e_s|=|t|$, and $|e_\phi|=n^{1/4}|\psi|$.

With this we have for $i=p,\ s,\ \phi$

$$\left|\int x\tilde{g}_i\right|\leqslant O\left(\frac{1}{n^2}\right)|e_i|\rightarrow|n^{1/2}\omega_i'(0)|\leqslant O(n^{-3/2})|e_i|$$

$$\left|\int x^2\tilde{g}_i\right|\leqslant O\left(\frac{1}{n^2}\right)|e_i|\rightarrow|n\omega_i''(0)|\leqslant O\left(\frac{1}{n}\right)|e_i|$$

$$\left|\int x^3\tilde{g}_i\right|\leqslant O\left(\frac{1}{n^{7/2}}\right)|e_i|\rightarrow|n^{3/2}\omega_i'''(0)|\leqslant O\left(\frac{1}{n^2}\right)|e_i|$$

$$\left|\int x^4\tilde{g}_i\right|\leqslant O\left(\frac{1}{n^4}\right)|e_i|\rightarrow|n^2\sup_u\omega_i''''(u)|\leqslant O\left(\frac{1}{n^2}\right)|e_i|$$

The same bounds remain true for the $\tilde{g}_i^0$.

Lemma 9. For sufficiently large n one has uniformly in n and in $|t| < (n/\hat{s}\rho_4)^{1/2}$ the following inequalities:

$$\text{(a)}\quad \left|\partial_j^t\left[\hat{v}_n(t) - \exp -\frac{\hat{s}t^2}{2}\right]\right| \leqslant O\left(\frac{1}{n}\right)\left(1 + t^{10}\right)\exp -\frac{\hat{s}t^2}{20}$$

$$\text{(b)}\quad \left|\partial_j^t\left\{\hat{\omega}_n^i(t) - \left(\exp -\frac{\hat{s}t^2}{2}\right)\left[\omega_i'(0)n^{1/2}t + \omega_i''(0)\frac{nt^2}{2}\right]\right\}\right|$$
$$\leqslant O(n^{-3/2})(1 + t^{14})\,|e_i|\exp -\frac{\hat{s}t^2}{20}$$

for $j = 0, 1, 2, 3, 4$ and $i = p, s, \phi$.

Proof. The proof follows exactly that of the case of symmetric functions. Observe that the bounds in Lemmas 5 and 6 are exactly the same if we take centered functions $\tilde{f}^0$ and $\tilde{g}^0$, because the translation is only $O(1/n^2)$. A sketch of the proof of Lemma 9 in the nonsymmetric case is given in Appendix A2.

For the exterior region we have the following.

Lemma 10 (Lemma 6.9 of [1]). For $|t| \geqslant (n/\rho_4\hat{s})^{1/2}$,

$$|F\tilde{f}^0(n^{1/2}t)| = |v(n^{1/2}t)| \leqslant \exp(-n^{-2/3})$$

The conditions for the application of the lemma are assured by Lemma 8.

As an immediate consequence of Lemmas 9 and 10 we have the following result.

Proposition 11. For n sufficiently large

(a) Given $\tilde{f} \in SS_1\mathbf{B} \subset \mathbf{K}$, define

$$\hat{f}(x) = \tilde{f}^{*n}(x)$$

then one has

$$n^{1/2}\hat{f}(n^{-1/2}x) = \frac{1}{(2\pi\hat{s})^{1/2}}\exp\left(-\frac{x^2}{2\hat{s}}\right) + \frac{1}{\hat{s}^{1/2}}\hat{\phi}\left(x + \frac{\hat{p}}{n^{1/2}}\right)$$

with $\hat{p} = n^2\tilde{J}$, $\hat{s} = n^2\tilde{\sigma}^2$, and $|\hat{\phi}| \leqslant O(1/n)$.

(b) If $\tilde{g}_i \in DSS_{1(p,s,\phi)}\mathbf{H} \subset \mathbf{K}$, then for

$$\mathring{g}_i(x) = \frac{1}{n}(\tilde{f}^0)^{*n-1} * \tilde{g}_i^0(n^{-1/2}x)$$

we have

$$|\mathring{g}_i(x) - n^{1/2}\,\partial_x h_{0,s}(x) - \tfrac{1}{2}n\,\partial_x^2 h_{0,s}(x)| \leqslant O(n^{-3/2})|e_i|$$

This completes the steps of the proof of Theorem 2.

APPENDIX

Throughout the appendix all the integrals are between $-\infty$ and $+\infty$ unless explicitly written otherwise.

A1. Some Technical Lemmas

We restate here some technical lemmas. Most of them were proven in Ref. 1.

Lemma A1. Let

$$T(x, y) = \operatorname{tgh}^{-1}(\operatorname{tgh} x \operatorname{tgh} y)$$
$$\delta(x, y) = \tfrac{1}{2} \ln[1 + \tau \operatorname{tgh}(x + y)], \qquad 0 \leqslant \tau \leqslant 1$$

Then for all $(x, y) \in \mathbf{R}$

(a) $|T(x, y)| \leqslant 5 \dfrac{|x|}{(1 + |x|)^{1/2}} \dfrac{|y|}{(1 + |y|)^{1/2}}$

(b) $|T(x, y)| \leqslant |xy|^{1/2}$

(c) $$|\delta(x, y)| \leqslant \frac{1}{2} |\ln(1 - |\tau \operatorname{tgh}(x + y)|)| \leqslant |T(\operatorname{tgh}^{-1} \tau, (x + y))|$$

Lemma A2. Define

$$A(f, g)(x) = (1 - \operatorname{tgh}^2 x) \int_\Omega \frac{dy}{|y| (1 - y^2)[1 - (\operatorname{tgh}^2 x)/y^2]} \times f(\operatorname{tgh}^{-1} y) f\left(\operatorname{tgh}^{-1} \frac{\operatorname{tgh} x}{y}\right)$$

with $\Omega = \{y, |\operatorname{tgh} x| < |y| < 1\}$. Assume $f, g \in \mathbf{L}_1 \cap \mathbf{L}_\infty$ and $\|f\|_1 = 1$. Then

$$\|A(f, g)\|_1 \leqslant \|g\|_1$$

$$\|A(f, g)\|_2 \leqslant O(1) \begin{cases} \|g\|_1 \|f\|_\infty + \|g\|_\infty \ln \|f\|_\infty & \text{if} \quad \|f\|_2 \geqslant 2 \\ (\|g\|_1 \|g\|_\infty)^{1/2} + \|g\|_1 \|f\|_\infty^{1/2} + \|g\|_\infty & \text{if} \quad \|f\|_\infty \leqslant 2 \end{cases}$$

This lemma was proven in Ref. 1 for f and g even. It can be easily extended to the general case if we consider separately the even and the odd parts of the functions involved.

Lemma A3. For every $\alpha > 0$ one has the inequalities

$$\|\phi\|_1 \leqslant 2\alpha \|\phi\|_\infty + \alpha^{-7/2} \|x^4\phi\|_2$$

$$\|\phi\|_1 \leqslant 2\alpha \|\phi\|_\infty + \alpha^{-5/2} \|x^3\phi\|_2$$

$$\|x\phi\|_1 \leqslant \alpha^2 \|\phi\|_\infty + \alpha^{-3/2} \|x^3\phi\|_2$$

$$\|x\phi\|_1 \leqslant \alpha^2 \|\phi\|_\infty + \alpha^{-5/2} \|x^4\phi\|_2$$

$$\|x^2\phi\|_1 \leqslant \alpha^3 \|\phi\|_\infty + \alpha^{-3/2} \|x^4\phi\|_2$$

Lemma A4. If for some $k \geqslant 1$, $\|x^{2k}f\|_2$ and $\|x^{2k}g\|_2$ exist and are finite, then

$$\begin{aligned}\|x^{2k}\tilde{g}\|_1 &= \int |T(x, y)|^{2k} \, |f(x)\, g(y)| \, dx \, dy \\ &\leqslant O(25^k) \, \|x^{2k}f\|_2 \, \|x^{2k}g\|_2\end{aligned}$$

Proof:

$$\begin{aligned}\|x^{2k}\tilde{g}\|_1 &\leqslant 5^{2k} \left[\int \frac{|x|^{2k}}{(1+|x|)^k} \, |f(x)| \, dx\right] \left[\int \frac{|y|^{2k}}{(1+|x|)^k} \, |g(y)| \, dy\right] \\ &\leqslant 25^k \, \|x^{2k}f\|_2 \, \|x^{2k}g\|_2 \left[\int \frac{dx}{(1+|x|)^{2k}}\right]^2\end{aligned}$$

A2. Bounds on the Moments

Consider

$$\tilde{J} = \int x\tilde{f}(x) \, dx = \iint T(x, y) \, f(x) \, f(y) \, dx \, dy$$

with

$$f(x) = h_{p/n, s/n}(x) + (n/s)^{1/2} \, \phi(n^{1/2}x)$$

We first estimate the Gaussian, i.e., the $\phi = 0$ contribution to $\tilde{J}$. For this we take the Taylor expansion of $T(x, y)$ around $(p/n, p/n)$ with the remainder at fourth order. This gives

$$\begin{aligned}\tilde{J}_G = \tilde{J}_0 - \frac{\tau^2}{2} \frac{n^2}{s} + \frac{1}{4!} \iint 4 \operatorname{tgh}^2 \left[2\frac{p}{n} + \Theta(x+y)\right] \\ - 3 \operatorname{tgh}^4 \left[2\frac{p}{n} + \Theta(x+y)\right] (x^4 + y^4 + 6x^2y^2) \\ \times \{4 \operatorname{tgh}^2[\Theta(x-y)]\} \, 4(x^3y + y^3x) \, h_{0,s}(x) \, h_{0,s}(y) \, dx \, dy\end{aligned}$$

where

$$\tilde{J}_0 = \text{tgh}^{-1}\, \text{tgh}^2(p/n), \qquad \tau = \text{tgh}\, 2J$$

and $0 \leqslant \Theta(z) \leqslant 1$ for all z. Note that $|\text{tgh}\, x| \leqslant |x|$ and each $|x|$ or $|y|$ contributes with $O(n^{-1/2})$ to the integral. Thus, the integral is bounded in modulus by $O(n^{-3})$.

To bound the non-Gaussian contribution observe that

$$T\left(x + \frac{p}{n}, y + \frac{p}{n}\right) = \tilde{J}_0 + T(x, y) + \delta(x, y)$$

If we write

$$\mathring{\phi}(x - p/n) = (n/s)^{1/2}\, \phi(n^{1/2} x)$$

we have

$$\tilde{J}_{\text{nG}} = 2 \iint \delta(x, y)\, h_{0,s}(x)\, \mathring{\phi}(y)\, dx\, dy$$

$$+ \iint [\delta(x, y) + T(x, y)]\, \mathring{\phi}(x)\, \mathring{\phi}(y)\, dx\, dy$$

But

$$\iint |T(x, y)|\, |\mathring{\phi}(x)|\, |\mathring{\phi}(y)|\, dx\, dy$$

$$\leqslant O(1)\, \|x\mathring{\phi}\|_1^2 = O(1/n)\, \|x\phi\|_1^2 \leqslant O(1/n)\, |\phi|^2$$

For the $\delta(x, y)$ part we write

$$\delta(x, y) = \tau(x + y) + r(x, y)(x + y)^2$$

with $|r(x, y)| \leqslant O(1/n)$. Since the first-order term does not contribute to the integral, we have

$$\left| \iint \delta(x, y)\, \mathring{\phi}(y)[h_{0,s}(x) + \mathring{\phi}(x)]\, dx\, dy \right|$$

$$\leqslant O(1/n) \iint (x + y)^2\, |\mathring{\phi}(y)|\, [|\mathring{\phi}(x)| + h_{0,s}(x)]\, dx\, dy$$

$$= O(1/n^2)[\|x^2\phi\|_1\, \|\phi\|_1 + \|x\phi\|_1^2 + \|\phi\|_1\, \|x\phi\|_1\, \|x^2\phi\|_1]$$

$$\leqslant O(1/n^2)(|\phi| + |\phi|^2)$$

But $|\phi| \leqslant (n^{1/4}/n)$ for $(0, 0, \phi) \in \mathbf{B}$, so

$$\tilde{J} = (p/n)^2 + O(n^{1/4}/n^3)$$

Let us now look at the first moment of $\tilde{g} = DSS_{1_{(p,s,\phi)}}(r, t, \psi)$. From the estimates above it is obvious that

$$\left| \int x\tilde{g}_\phi(x)\, dx \right| = \left| \iint T(x, y)\, f(x)\, g_\phi(y)\, dx\, dy \right|$$
$$\leqslant \mathrm{O}(n^{1/4}/n^2)\, |\psi|$$

Consider

$$\int x\tilde{g}_p(x)\, dx = \frac{r}{s} \iint T(x, y) \left[h_{p/n, s/n}(x) + \mathring{\phi}\left(x - \frac{p}{n} \right) \right]$$
$$\times \left[\left(y - \frac{p}{n} \right) h_{p/n, s/n}(y) \right] dx\, dy$$

The Gaussian part gives us

$$\frac{r}{s} \iint \left[\frac{1}{2} \tau y + R(x, y)(x + y)^3 \right] y h_{0,s}(x)\, h_{0,s}(y)\, dx\, dy$$

with

$$|R(x, y)| \leqslant |x| + |y| + (|x| + |y|)^3, \qquad \tau \sim 2p/n$$

Thus

$$\int x\tilde{g}_p(x)\, dx = r \frac{p}{n^2} + O\left(\frac{1}{n^2 n^{1/2}} \right)$$
$$+ \frac{r}{s} \iint T\left(x + \frac{p}{n}, y + \frac{p}{n} \right) y\mathring{\phi}(x)\, h_{0,s}(y)\, dx\, dy$$

The last term on the right-hand side is bounded by

$$|r| \iint \left| x + \frac{p}{n} \right| |y|\, |\mathring{\phi}(x)|\, h_{0,s}(y)\, dx\, dy \leqslant |r|\, O\left(\frac{1}{n^{3/2}} \right) |\phi|$$

Finally, consider

$$\left| \int x\tilde{g}_s(x)\, dx \right| = \frac{|t|}{2s} \left| \iint T(x, y) \left\{ \left[\frac{n}{s} \left(y - \frac{p}{n} \right)^2 - 1 \right] h_{p/n, s/n}(y) \right. \right.$$
$$\left. \left. + \mathring{\phi}\left(y - \frac{p}{n} \right) \right\} \left[h_{p/n, s/n}(x) + \mathring{\phi}\left(x - \frac{p}{n} \right) \right] dx\, dy \right|$$

The Gaussian contribution to it is $|t|\,O(1/n^3)$ and the rest is given by

$$\iint T\left(x+\frac{p}{n},\, y+\frac{p}{n}\right)\left[\left(\frac{n}{s}\,y^2-1\right)h_{0,s}(y)\,\mathring{\phi}(x)\right.$$
$$\left.+\,h_{0,s}(x)\,\mathring{\phi}(y)+\mathring{\phi}(x)\,\mathring{\phi}(y)\right]dx\,dy$$

This was already bounded in $\tilde{J}_{\mathrm{nG}}$ by $O(n^{1/4}/n^3)$.

We now study the second moments. The squared variance after summing over one spin is given by

$$\tilde{\sigma}^2=\int (x-\tilde{J})^2\,\tilde{f}(x)\,dx=\iint [T(x,y)-\tilde{J}]^2\,f(x)\,f(y)\,dx\,dy$$

We write $[T(x,y)-\tilde{J}]$ as $[T(x,y)-\tilde{J}_0]-\Delta$, where $|\Delta|=|\tilde{J}_0-\tilde{J}|<O(1/n^2)$. It is easy to show in the same way as for $\tilde{J}$ that the Gaussian contribution to the squared variance is given by

$$\tilde{\sigma}^2_{\mathrm{G}}=s^2/n^2+O(1/n^3)$$

The non-Gaussian part is

$$\begin{aligned}
|\tilde{\sigma}^2_{\mathrm{nG}}|&=\left|\iint [T(x,y)+\delta(x,y)+\Delta]^2\right.\\
&\qquad\left.\times[2h_{0,s}(x)\,\mathring{\phi}(y)+\mathring{\phi}(x)\,\mathring{\phi}(y)]\,dx\,dy\right|\\
&\leqslant O(n^{1/4}/n^5)+\left|\iint [T(x,y)+\delta(x,y)]^2\right.\\
&\qquad\left.\times[2h_{0,s}(x)+\mathring{\phi}(x)]\,\mathring{\phi}(y)\,dx\,dy\right|\\
&\leqslant O(1)\iint (|xy|+\tau|x|+\tau(y|)^2\\
&\qquad\times[2h_{0,s}(x)+|\mathring{\phi}(x)|]\,|\mathring{\phi}(y)|\,dx\,dy+O(n^{1/4}/n^5)\\
&\leqslant O(1/n^2)\,|\phi|\leqslant O(n^{1/4}/n^3)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|\int x^2\tilde{g}_p(x)\,dx\right|\\
&\quad=\frac{|r|}{s}\iint T^2\left(x+\frac{p}{n},\, y+\frac{p}{n}\right)[h_{0,s}(x)+\mathring{\phi}(x)]\;yh_{0,s}(y)\,dx\,dy
\end{aligned}$$

is easily bounded by $|r|\,O(1/n^{2+1/2})$. We also have

$$\left|\int x^2 \tilde{g}_\phi(x)\,dx\right| \leqslant O(1/n^2)\,|\psi|$$

Consider now

$$\int x^2 \tilde{g}_s(x)\,dx = \frac{t}{s}\iint T^2\left(x+\frac{p}{n},\ y+\frac{p}{n}\right)[h_{0,s}(x)+\mathring{\phi}(x)]$$

$$\times\left[\left(\frac{n}{s}\,y^2-1\right)h_{0,s}(y)+\mathring{\phi}(y)\right]dx\,dy$$

The non-Gaussian part is bounded as for $\tilde{\sigma}^2$ by $|t|\,O(n^{1/4}/n^3)$. It is easy to see that the leading term of the Gaussian part is t/n^2 and the rest is $O(1/n^3)\,|t|$.

Finally, let us study the third moment of $\tilde{f}$:

$$\tilde{\mu}_3 = \iint [T(x,y)-\tilde{J}]^3 f(x)\,f(y)\,dx\,dy$$

$$= \iint T^3(x,y)\,f(x)\,f(y)\,dx\,dy - \tilde{J}^3 - 3\tilde{J}\tilde{\sigma}^2$$

where the constant term is $O(1/n^4)$. We have

$$\tilde{\mu}_{3,\mathrm{G}} = \iint [\tilde{J}_0 + a(x+y) + b_1(x^2+y^2) + b_2 xy$$

$$+\ c(x,y)(x+y)^3]^3\,h_{0,s}(x)\,h_{0,s}(y)\,dx\,dy$$

where $\tilde{J}_0 = O(1/n^2), a = O(1/n), b_1 = O(1/n^2)$, $b_2 = O(1)$, and

$$|c(x,y)| \leqslant |x| + |y| + (|x|+|y|)^3$$

So the leading term is given by the integral of $\tilde{J}_0 b_2^2 x^2 y^2$ and is also $O(1/n^4)$.

For the $\phi\phi$ part of $\tilde{\mu}_3$ we have, by Lemma A4,

$$\left|\frac{n}{s}\iint T^3(x,y)\,\phi(n^{1/2}x)\,\phi(n^{1/2}y)\,dx\,dy\right|$$

$$\leqslant O\left(\frac{1}{n^{5/2}}\right)\|x^3\phi\|_2^2 \leqslant O\left(\frac{1}{n^4}\right)$$

To evaluate the mixed $h\phi$ contribution observe that

$$2 \iint T^3(x, y)\, h_{p/n,s/n}(x)\, \mathring{\phi}(y + pn)\, dx\, dy$$

$$= 2 \iint [T(x, y) + \delta(x, y) + \tilde{J}_0]^3\, h_{0,s}(x)\, \mathring{\phi}(y)\, dx\, dy$$

$$= 2 \iint [3T^2(x, y)\, \delta(x, y) + 3T(x, y)\, \delta^2(x, y) + \delta^3(x, y)]$$

$$\times h_{0,s}(x)\, \mathring{\phi}(y)\, dx\, dy + O(1/n^{1/4})$$

$$\leqslant O(1/n^3 n^{1/4})\, |\phi|$$

In the same way

$$\left| \int x^3 \tilde{g}_\phi(x)\, dx \right| \leqslant O\left(\frac{1}{n^3 n^{1/4}}\right) |\psi|$$

$$\left| \int (x - \tilde{J})^3\, \tilde{g}_s(x)\, dx \right| \leqslant O\left(\frac{1}{n^4}\right) |t|$$

$$\left| \int (x - \tilde{J})^3\, \tilde{g}_p(x)\, dx \right| \leqslant \frac{|r|}{s} \left| \int T^3(x, y) \left[h_{p/n,s/n}(x) + \mathring{\phi}\left(x + \frac{p}{n}\right) \right] \right.$$

$$\left. \times \left(y - \frac{p}{n} \right) h_{p/n,s/n}(y)\, dx\, dy \right|$$

$$\leqslant O\left(\frac{1}{n^4}\right) |r|$$

A3. Proof of Lemma 9a for $j = 0$

Let us write $\tau = n^{1/2} t$. We can write $v(\tau)$ as

$$v(\tau) = 1 - \frac{t^2}{2} \frac{\hat{s}}{n} - i \frac{t^3}{6} \rho_3 \left(\frac{\hat{s}}{n}\right)^{3/2} + t^4 r(t)\, \rho_4 \left(\frac{\hat{s}}{n}\right)^2$$

with $|r(t)| \leqslant O(1)$.

If we denote

$$\alpha(t) = \frac{i}{3} \left(\frac{\hat{s}}{n}\right)^{1/2} \rho_3 t - 2 \frac{\hat{s}}{n} \rho_4 t^2 r(t)$$

then

$$v(\tau) = 1 - \frac{t^2}{2} \frac{\hat{s}}{n} [1 + \alpha(t)]$$

If $|t| \leqslant (n/\rho_4 \hat{s})^{1/2}$ and using the bounds on ρ_3 and ρ_4, we have that

$$|\alpha(t)| < \tfrac{1}{12} + O(1/n)$$

For n sufficiently large and using

$$\ln(1-x) = -x - \int_0^x t \frac{dt}{1-t}$$

we obtain

$$\ln v(\tau) = -\frac{t^2}{2} \frac{\hat{s}}{n} - \frac{i}{6} \rho_3 \left(\frac{\hat{s}}{n}\right)^{3/2} t^3 + \left(\frac{\hat{s}}{n}\right)^2 t^4 \rho_4 q(t)$$

with $|q(t)| < \frac{2}{5}$. So we have

$$\hat{v}_n(t) = [v(\tau)]^n = \exp\left(-\frac{\hat{s}t^2}{2}\right) \exp\left[-\frac{i}{6}\rho_3 \frac{\hat{s}^{3/2}}{n^{1/2}} t^3 + \frac{\hat{s}^2}{n} t^4 \rho_4 q(t)\right]$$

Using the inequality $|e^x - 1| \leqslant |x| e^{|x|}$, we obtain

$$\left|\hat{v}_n(t) - \exp -\frac{\hat{s}t^2}{2}\right| \leqslant O\left(\frac{1}{n}\right)(1+t^4) \exp -\frac{\hat{s}t^2}{2}$$

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